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Potential symmetries and solutions by reduction of partial differential equations

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Abstract. We determine some necessary conditions for a given partial differential equation $\mathscr{E}$, written in conservative form to admit a potential symmetry (PS). A PS of $\mathscr{E}$ is a point symmetry of the auxiliary system $\mathscr{S}_p$ obtained introducing a potential as further unknown function, then a PS leads to the construction of solutions via the classical reduction method. Given a PS, we introduce an algorithm that allows us to determine a class of $\mathscr{E}$-solutions which includes the ones obtained as invariant solutions under the related point symmetry of $\mathscr{S}_p$. As examples, we consider a Fokker–Planck equation, a wave equation in non-homogeneous media and a quasilinear wave equation.

1. Introduction

For a partial differential equation (PDE) $\mathscr{E}$ in two variables, the reduction procedure is based on the use of a similarity variable, which allows to get solutions of the original PDE $\mathscr{E}$ by integration of an ordinary differential equation (ODE).

Reduction procedures are used to search solutions which are invariant under local symmetries, classical and weak [1, 2]. In [3], it is proved that the reduction procedure, which uses *non-classical symmetries,* includes the direct reduction method, defined in [4, 5].

In [6], Bluman and others suggested a method to find a new class of symmetries for a PDE $\mathscr{E}$, in case it is written in conservative form. They analysed the Lie symmetries of the system $\mathscr{S}_p$ that is obtained introducing a potential as further unknown function. Any group $\mathscr{G}_S$ of Lie transformations for $\mathscr{S}_p$ induces a symmetry for $\mathscr{E}$; when at least one of the generators of $\mathscr{G}_S$ (associated to the variables and the unknown function of $\mathscr{E}$) depends explicitly on the potential, then the corresponding symmetry of $\mathscr{E}$ is neither a point nor a Lie–Bäcklund symmetry. These new symmetries of $\mathscr{E}$ are called potential symmetries.

In this paper, we determine some necessary conditions for an equation, written in conservative form, to admit potential symmetries.

We prove that the only equations which can admit potential symmetries are those in which either the flux or the density are functions depending at most on the first derivatives of the unknown function. A further examination specifies the possible forms and gives a characterization of some properties of generators.

The potential symmetries of $\mathscr{E}$ being point symmetries of the system $\mathscr{S}_p$, can be determined by Lie's algorithm; this fact makes the potential symmetries useful in looking for solutions of $\mathscr{E}$ using a reduction method. We can compute the solutions $\mathscr{F}$ for the invariant surface conditions of the group $\mathscr{G}_S$. The invariant solutions of $\mathscr{S}_p$

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are the solutions $\mathscr{F}$ which are also solutions of $\mathscr{S}_p$. These solutions will give naturally a class, $\mathscr{F}_E$, of solutions for $\mathscr{E}$ [7]. In $\mathscr{F}$ there may exist also solutions for $\mathscr{E}$ which are not in $\mathscr{F}_E$; these are determined by direct substitution of $\mathscr{F}$ in $\mathscr{E}$.

As examples we consider a Fokker–Planck equation, a wave equation in non-homogeneous media and a quasilinear wave equation.

2. Conservative forms

A PDE $\mathscr{E}$ of order n, in the unknown function $u(x, t)$ is written in a conservative form if it has the form

$$D_t F - D_x G = 0. \tag{2.1}$$

Here, the density F and the flow G are functions of $(x, t, u, \underset{1}{u}, \ldots, \underset{n-1}{u})$, $\underset{k}{u}$ stands for the set of kth order derivatives of u, while D_x and D_t are the operators of total derivation wrt x and t. Clearly, a PDE can be written in conservative form (2.1) only if it is quasilinear.

If it possible to define a density F and a flow G and write a PDE in the form (2.1), then the same can be done with an infinite number of other densities and flows, related to the first by

$$\tilde{F} = F + D_x \tilde{H} \qquad \tilde{G} = G + D_t \tilde{H} \tag{2.2}$$

with an arbitrary regular function $H(x, t, u, \underset{1}{u}, \ldots, \underset{k}{u}, \ldots)$.

There could also exist, for certain equations, several flows and densities which are not related by the relation (2.2).

For example the Fokker–Planck equation (where we have set physical coefficients to be unity)

$$E_1 \equiv u_t - u_{xx} - xu_x - u = 0 \tag{2.3}$$

may be written in conservative form with the following two choices of flows and densities

$$F_1 = u \qquad G_1 = u_x + xu \tag{2.4a}$$

and

$$F_2 = u \int \exp(x^2/2)\, \mathrm{d}x \qquad G_2 = (u_x + xu) \int \exp(x^2/2)\, \mathrm{d}x - u \exp(x^2/2). \tag{2.4b}$$

Similarly, for the wave equation in non-homogeneous media (where we have set physical coefficients to be unity)

$$E_2 \equiv u_{tt} - xu_{xx} = 0 \tag{2.5}$$

there exist the following two choices

$$F_1 = u_t / x \qquad G_1 = u_x \tag{2.6a}$$

and

$$F_2 = u_t \qquad G_2 = xu_x - u. \tag{2.6b}$$

By considering a potential $v(x, t)$ as an auxiliary unknown function, the following system $\mathscr{S}_p$ can be associated with (2.1):

$$v_x = F \qquad v_t = G. \tag{2.7}$$

The forms (2.2) are equivalent in the following sense: the corresponding systems $\mathcal{S}_p$ are the same, if the definition of v only is changed. On the other hand, densities and flows of the form (2.4) or (2.6) are not equivalent. Hence, when a PDE can be written in conservative form in more than one way (not equivalent), it is essential to specify which one is the considered conservative form when we are looking for potential symmetries. In the last section, we will see that (2.5) admits potential symmetries when written in conservative form by means of F_1 and G_1; on the contrary, there are no such symmetries if it is assumed that the conservative form makes use of F_2 and G_2.

3. Potential symmetries

A point symmetry group for $\mathcal{S}_p$ is defined by the following equations:

$$x' = x + \varepsilon\xi(x, t, u, v) + O(\varepsilon^2) \qquad t' = t + \varepsilon\tau(x, t, u, v) + O(\varepsilon^2)$$

$$u' = u + \varepsilon\eta(x, t, u, v) + O(\varepsilon^2) \qquad v' = v + \varepsilon\phi(x, t, u, v) + O(\varepsilon^2)$$

and it is completely determined by the generators ξ, τ, η, ϕ. Point symmetries, which verify $\xi_v^2 + \tau_v^2 + \eta_v^2 = 0$, correspond to point symmetries of $\mathscr{E}$. Instead, we obtain potential symmetries of $\mathscr{E}$, if $\xi_v^2 + \tau_v^2 + \eta_v^2 > 0$.

It is well known that the homogeneous linear system which characterizes the generators is obtained from

$$\mathscr{H}^{(n-1)}(v_x - F)|_{\mathcal{S}_p} = 0 \qquad \mathscr{H}^{(n-1)}(v_t - G)|_{\mathcal{S}_p} = 0 \tag{3.1}$$

which must hold identically. Here $\mathscr{H}^{(n-1)}$ is the operator

$$\mathscr{H}^{(n-1)} = \xi\frac{\partial}{\partial x} + \tau\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial u} + \phi\frac{\partial}{\partial v}$$

$$+ \sum_{r+s=1}\left(U_{rs}^{(1)}\frac{\partial}{\partial u_{rs}} + V_{rs}^{(1)}\frac{\partial}{\partial v_{rs}}\right) + \ldots + \sum_{r+s=n-1}\left(U_{rs}^{(n-1)}\frac{\partial}{\partial u_{rs}} + V_{rs}^{(n-1)}\frac{\partial}{\partial v_{rs}}\right).$$

Here

$$u_{rs} = \frac{\partial^{r+s}u}{\partial x^r \partial t^s} \qquad v_{rs} = \frac{\partial^{r+s}v}{\partial x^r \partial t^s} \qquad r, s \in \mathbb{N}_0$$

and $U_{rs}^{(k)}$ and $V_{rs}^{(k)}$ are extensions of order $k = r + s$ (r times in x and s times in t) of η and ϕ, respectively. Observe that $U_{rs}^{(k)}$ depends on the derivatives of v of order $r + s$ in the following way:

$$U_{rs}^{(k)} = v_{rs}(\xi_v u_x + \tau_v u_t - \eta_v) + \hat{U}_{rs}^{(k)}$$

where $\hat{U}_{rs}^{(k)}$ does not depend on the derivatives of v of order $k = r + s$.

Let $\mathscr{E}$ be of order $n > 2$; clearly, at least one, either F or G, depends on the $(n-1)$st derivative; say G. Hence we have that

$$\mathscr{H}^{(n-1)}(v_t - G) \equiv V_{01}^{(1)} - \mathscr{H}^{(n-1)}G \equiv \sum_{r+s=n-1}\left(\frac{\partial G}{\partial u_{rs}} v_{rs}\right)(\xi_v u_x + \tau_v u_t - \eta) + \ldots \tag{3.2}$$

where the given terms are the only ones that depend on the $(n-1)$st-order derivatives of v.

Let F be of order $h \leqslant n-1$, that is $h = n-1-\bar{h}$ for some $\bar{h} \in \mathbb{N}_0$. If we want to evaluate (3.2) on the manifold $\mathcal{S}_p$, besides (2.7), we have to consider the differential consequences of $v_x - F = 0$, up to order $\bar{h}$; then, we have to substitute all derivatives of v in (3.2) up to order $\bar{h}+1$, but the pure derivatives in t of order greater than 1. Such derivatives act on the explicit term in (3.2) only if $\bar{h}+1 = n-1$, that is only if $h=1$. Then, the following theorem may be proved.

Theorem 1. The necessary conditions for (2.1), of order $n>2$, to admit potential symmetries is that

$$\frac{\partial G}{\partial u_{0,n-1}} = 0 \qquad \text{and} \qquad F = F(x, t, u, u_x, u_t). \tag{3.3}$$

(Clearly, an analogous theorem holds if we change G to F and t to x in (3.3).)

Since for $n=2$, $F = F(x, t, u, u_x, u_t)$ and $G = G(x, t, u, u_x, u_t)$, we have shown that potential symmetries can exist only if the density or the flow depends at most on the first derivatives of u; now, we want to get a better characterization of this dependence and of the structure of the generators.

Assume the first equation of $\mathcal{S}_p$ in the form

$$v_x = F(x, t, u, u_x, u_t). \tag{3.4}$$

The first equation in (3.1) is verified if and only if

$$\tau_x + \tau_u u_x + \tau_v F - \frac{\partial F}{\partial u_t}(\xi_v u_x + \tau_v u_t - \eta_v) = 0 \tag{3.5}$$

$$\begin{aligned} &\frac{\partial F}{\partial u_x}[\eta_x + \eta_u u_x - u_x(\xi_x + \xi_u u_x) - u_t(\tau_x + \tau_u u_x) - F(\xi_v u_x + \tau_v u_t - \eta_v)] \\ &\qquad + \frac{\partial F}{\partial u_t}[\eta_t + \eta_u u_t - u_x(\xi_t + \xi_u u_t) - u_t(\tau_t + \tau_u u_t)] + \xi\frac{\partial F}{\partial x} + \tau\frac{\partial F}{\partial t} + \eta\frac{\partial F}{\partial u} + F^2\xi_v \\ &\qquad + F(\xi_x + \xi_u u_x - \phi_v) - \phi_x - \phi_u u_x = 0. \end{aligned} \tag{3.6}$$

Equation (3.5) implies that (3.4) must be of the form

$$v_x = H(x, t, u, u_x)u_t + K(x, t, u, u_x). \tag{3.7}$$

Hence (3.5) and (3.6) satisfy the following relations:

$$\tau_v H \frac{\partial H}{\partial u_x} = 0 \tag{3.8}$$

$$\frac{\partial H}{\partial u_x}[K\tau_v + \tau_x - H(\eta_v - \xi_v u_x)] + \tau_v H \frac{\partial K}{\partial u_x} - H^2\xi_v + H\tau_u = 0 \tag{3.9}$$

$$\begin{aligned} &\frac{\partial H}{\partial u_x}[K(\xi_v u_x - \eta_v) + \xi_u u_x^2 + u_x(\xi_x - \eta_u) - \eta_x] + 2\frac{\partial K}{\partial u_x}(K\tau_v + \tau_x + \tau_u u_x) \\ &\qquad + H(\xi_x - \tau_t + \eta_u - \phi_v) - \xi\frac{\partial H}{\partial x} - \tau\frac{\partial H}{\partial t} - \eta\frac{\partial H}{\partial u} + 2HK\xi_v = 0 \end{aligned} \tag{3.10}$$

$$H(\xi_v u_x - \eta_v) - K\tau_v - \tau_x - \tau_u u_x = 0 \tag{3.11}$$

$$\frac{\partial K}{\partial u_x}[\eta_x - \xi_u u_x^2 - u_x(\xi_x - \eta_u)] + K\frac{\partial K}{\partial u_x}(\eta_v - \xi_v u_x) + \xi\frac{\partial K}{\partial x} + \tau\frac{\partial K}{\partial t} + \eta\frac{\partial K}{\partial u}$$
$$+ K^2\xi_v + K(\xi_x + \xi_u u_x - \phi_v) + H(\eta_t - \xi_t u_x) - \phi_u u_x - \phi_x = 0. \tag{3.12}$$

For (3.8), there are the following possibilities:

(1) $H = H(x, t, u)$ $\quad \tau_v \neq 0$

(2) $\tau_v = 0$.

Case (1).
From (3.11), it follows that

$$K = \frac{u_x(H\xi_v - \tau_u) - (\tau_x + H\eta_v)}{\tau_v}$$

which verifies (3.9) identically. Equations (3.10) and (3.12) give relations between H and the generators.

Case (2).
From (3.11) it follows that $H = (\tau_u u_x + \tau_x)/(\xi_v u_x - \eta_v)$ which is compatible with (3.9) if and only if

(2a) $H = \tau_u/\xi_v$ $\quad \xi_v \neq 0$ $\quad \tau_v \neq 0$ $\quad \tau_x\xi_v + \tau_u\eta_v = 0$

(2b) $H = -\tau_x/\eta_v$ $\quad \eta_v \neq 0$ $\quad \tau_x \neq 0$ $\quad \xi_v = \tau_u = 0$

(2c) $H = 0$ $\quad \tau_x = \tau_u = 0$.

In cases (2a) and (2b), from (3.10) we may infer that ξ, η, ϕ are at most linear in v and that K is linear in u_x. Equations (3.10) and (3.12) establish relations between the generators and the coefficients of that linear form. In case (2c), we have that (3.9), (3.10), (3.11) are identically verified, while (3.12) assumes the simplified form that is obtained by setting $H = 0$, and it defines K.

In conclusion we have proved the following:

Theorem 2. Equation (2.1) admits symmetry potentials only if equation (3.4) assumes one of the following forms:

$$v_x = H(x, t, u)u_t + K_1(x, t, u)u_x + K_2(x, t, u) \tag{3.13}$$

where $H \neq 0$; otherwise

$$v_x = K(x, t, u, u_x) \tag{3.14}$$

and in this case is $\tau = \tau(t)$.

If K in (3.14) is a polynomial of degree n in u_x from (3.12), when $n \geqslant 3$ (i.e. $2n-1 > n+1$) it follows that $\xi_v = \eta_v = 0$. Therefore we have:

Theorem 3. If K in (3.14) is a polynomial of degree n in u_x, there exist potential symmetries only if $n \leqslant 2$.

4. Solutions by reduction

Let $\mathscr{E}$ be a partial differential equation which can be written in conserved form with a choice of F and G which satisfies the necessary conditions of the last section. We suppose to have determined a potential symmetry of $\mathscr{E}$. Now it is interesting to clarify how it is possible to use these symmetries to find exact solutions by reduction methods.

Given a point symmetry for $\mathscr{S}_p$, the invariant surface conditions are

$$\xi(x, t, u, v)u_x + \tau(x, t, u, v)u_t - \eta(x, t, u, v) = 0 \tag{4.1}$$

$$\xi(x, t, u, v)v_x + \tau(x, t, u, v)v_t - \phi(x, t, u, v) = 0. \tag{4.2}$$

The solutions of the associated characteristic system are given by three independent integrals

$$s_1(x, t, u, v) = c_0 \qquad s_2(x, t, u, v) = c_1 \qquad s_3(x, t, u, v) = c_2 \tag{4.3}$$

with $\partial(s_1, s_2, s_3)/\partial(u, v)$ of rank 2.

The solutions of (4.1) and (4.2) are defined as one-parameter families of characteristic curves (4.3). If we assume $c_0 = z$ as parameter and $c_1 = h_1(z)$, $c_2 = h_2(z)$ from (4.3) we obtain

$$u = U(x, t, z, h_1(z), h_2(z)) \tag{4.4}$$

$$v = V(x, t, z, h_1(z), h_2(z)) \tag{4.5}$$

$$G(x, t, z, h_1(z), h_2(z)) = 0. \tag{4.6}$$

The last equation defines implicitly the similarity variable z as function of (x, t). We point out that (4.4) is a family of solutions of the second-order equation

$$\bar{\eta}(x, t, u, \underset{1}{u}, \underset{2}{u}) = 0 \tag{4.7}$$

that is obtained by eliminating v between (4.1) and (4.2).

The invariant solutions of $\mathscr{S}_p$ are given by (4.4) and (4.5) where the $h_i(z)$ are the solutions of the ordinary system $\bar{\mathscr{S}}$, which is obtained by substitution in $\mathscr{S}_p$. $\mathscr{E}$ being a differential consequence of $\mathscr{S}_p$, the solutions of $\mathscr{S}_p$ give those solutions $\mathscr{F}_E$ of $\mathscr{E}$, which verify the differential relation

$$\tilde{\eta}(x, t, u, \underset{1}{u}, \ldots, \underset{n-1}{u}) = 0 \tag{4.8}$$

obtained by eliminating v between (4.1) and

$$\xi F + \tau G - \phi = 0. \tag{4.9}$$

We can determine a family $\mathscr{F}_E^*$ of $\mathscr{E}$-solutions by direct introduction of (4.4) and (4.6) in $\mathscr{E}$. We obtain, in this way, a relation involving z, h_1, h_2, the derivative up to order n and one parameter given by the x or the t. By imposing that the relation is identically zero for any value of the parameter, this will result in an ordinary system $\bar{\mathscr{S}}$ on the $h_i(z)$. $\mathscr{F}_E^*$ is given by (4.4) where $h_i(z)$ are solutions of $\bar{\mathscr{S}}$; then $\mathscr{F}_E^*$ is a family of solutions for (4.7). On the other hand $\mathscr{F}_E$, besides (4.7), verifies also (4.8), then $\mathscr{F}_E$ is enclosed in $\mathscr{F}_E^*$. In [7] and [8] potential symmetries have been applied to obtain only the solutions $\mathscr{F}_E$. Here we apply potential symmetries to obtain the wider class $\mathscr{F}_E^*$ of $\mathscr{E}$-solutions.

5. Examples

In this section we clarify the generalization of the classical reduction method above introduced by some examples.

Example 1.
For the Fokker–Plank equation (2.3), if we consider the corresponding system

$$v_x = u \qquad v_t = u_x + xu \tag{5.1}$$

which is a particular case of (3.14), we obtain point symmetries with the following generators:

$$\begin{array}{lllll}
\mathscr{X}_1: & \xi = -x\exp(2t) & \tau = -\exp(2t) & \eta = (ux^2 + 2xv + 2u)\exp(2t) & \\
 & \phi = v(x^2+1)\exp(2t) & & & \\
\mathscr{X}_2: & \xi = \exp(t) & \tau = 0 & \eta = -(ux+v)\exp(t) & \phi = -vx\exp(t) \\
\mathscr{X}_3: & \xi = \exp(-t) & \tau = \eta = \phi = 0 & & \\
\mathscr{X}_4: & \xi = -x\exp(-2t) & \tau = \exp(-2t) & \eta = u\exp(-2t) & \phi = 0 \\
\mathscr{X}_5: & \tau = 1 & \xi = \eta = \phi = 0 & & \\
\mathscr{X}_6: & \xi = \tau = 0 & \eta = u & \phi = v &
\end{array}$$

and ∞-dimensional symmetry, which is a consequence of linearity. In all the symmetries above, only $\mathscr{X}_1$ and $\mathscr{X}_2$ are potential symmetries for (2.3).

For the potential symmetry $\mathscr{X}_1$ the characteristic system related to the invariant surface conditions

$$xu_x + u_t + ux^2 + 2vx + 2u = 0 \tag{5.2}$$

$$xv_x + v_t + v(x^2+1) = 0 \tag{5.3}$$

admits the following three integrals

$$c_0 = x^{-1}\exp(t) \qquad c_1 = vx\exp(x^2/2) \qquad c_2 = (ux^2 + vx^3)\exp(x^2/2). \tag{5.4}$$

Then the solutions of (5.2), (5.3) are

$$\begin{aligned} u &= (h_2(z)x^{-2} - h_1(z))\exp(-x^2/2) \\ v &= h_1(z)x^{-1}\exp(-x^2/2) \qquad z = x^{-1}\exp(t). \end{aligned} \tag{5.5}$$

Here the equation (4.7) reads

$$\bar{\eta} \equiv u_{tt} + 2xu_{xt} + x^2u_{xx} + (2x^2+2)u_t + (2x^3+3x)u_x + (x^4+4x^2)u = 0. \tag{5.6}$$

To find the solutions $\mathscr{F}_E^*$, we introduce $(5.5)_1$ in (2.3) obtaining

$$z^2(h_2''z^2 + 6h_2'z + 6h_2) + \exp(2t)(2h_2 - 2zh_1' - z^2h_1'') = 0$$

which must hold for any value of t. In this way we have the system $\bar{\mathscr{S}}$:

$$h_2''z^2 + 6h_2'z + 6h_2 = 0$$

$$h_1''z^2 + 2h_1'z - 2h_2 = 0.$$

The family $\mathscr{F}_E^*$ is therefore:

$$u = [a_1(1-x^2) - a_2x\exp(t) + b_1(3x - x^3)\exp(-t) + b_2\exp(2t)]\exp(-2t - x^2/2) \tag{5.7}$$

where a_i, b_i are arbitrary constants.

The family $\mathcal{F}_E$ is given by the functions (5.7) which are also solutions of

$$\tilde{\eta} \equiv (x^3 - x)u_x + (x^2+1)u_t + u(2 - x^2 + x^4) = 0 \tag{5.8}$$

obtained by setting $b_1 = b_2 = 0$.

We point out that the (5.6) can be viewed, in a formal way, as an invariant surface condition of a second-order generalized symmetry with evolutionary operator [1]

$$\bar{\mathscr{X}} = \bar{\eta}\frac{\partial}{\partial u}. \tag{5.9}$$

We can check that

$$\bar{\mathscr{X}}^{(2)}E_1 \equiv \lambda_{00}E_1 + \lambda_{10}D_xE_1 + \lambda_{01}D_tE_1 + \lambda_{20}D_{xx}E_1 + \lambda_{11}D_{xt}E_1 + \lambda_{02}D_{tt}E_1 + \mu\bar{\eta} \tag{5.10}$$

where the Lagrange multipliers are

$$\lambda_{00} = x^4 + 8x^2 + 8 \qquad \lambda_{10} = 2x^3 + 7x \qquad \lambda_{01} = 2x^2 + 6$$

$$\lambda_{20} = x^2 \qquad \lambda_{11} = 2x \qquad \lambda_{02} = 1 \qquad \mu = -4.$$

The relation (5.10) suggests interpreting the generalized symmetry (5.9) as a *non-classical generalized symmetry* for (2.3), in accordance with the definition of non-classical point symmetry.

On the other hand the relation (5.8), which is linear in u_x and u_t, is the invariant surface condition of the non-classical point symmetry of (2.3), with operator

$$\tilde{\mathscr{X}} = \frac{x^3 - x}{x^2+1}\frac{\partial}{\partial x} + \frac{\partial}{\partial t} + u\frac{x^2 - x^4 - 2}{x^2+1}\frac{\partial}{\partial u}.$$

By the same procedure we find that the family $\mathcal{F}_E^*$ related to $\mathscr{X}_2$ is given by

$$u = (c_1 - c_2\exp(-t)x)\exp(-x^2/2).$$

From this family the $\mathcal{F}_E$ solutions are obtained by setting $c_1 = 0$.

The second conservative form (2.4), which is again a particular case of (3.14), gives the system $\hat{\mathscr{S}}_p$

$$v_x = u\int \exp(x^2/2)\,\mathrm{d}x \qquad v_t = (u_x + xu)\int \exp(x^2/2)\,\mathrm{d}x - u\exp(x^2/2). \tag{5.11}$$

It admits only the point symmetries corresponding to the generators

$$\hat{\mathscr{X}}_1: \qquad \tau = 1 \qquad \xi = \eta = \phi = 0$$

$$\hat{\mathscr{X}}_2: \qquad \xi = \tau = 0 \qquad \eta = u \qquad \phi = v$$

besides the ∞-dimensional symmetry. Clearly none of those symmetries is potential.

Example 2.

For the wave equation in non-homogeneous media (2.5), if we consider the corresponding system $\mathscr{S}_p$

$$v_x = u_t/x \qquad v_t = u_x \tag{5.12}$$

which is a particular case of (3.13) ($H = 1/x$, $K_1 = K_2 = 0$), we obtain only one potential symmetry [7], i.e. the point symmetry of (5.12) associated with the generators

$$\mathscr{X}: \qquad \xi = 4xt \qquad \tau = 4x + t^2 \qquad \eta = -2vx \qquad \phi = -2tv - 2u. \tag{5.13}$$

The solutions of the invariant surface conditions of the vector field (5.13) are

$$u = tx^{-1/4}h_1(z) - 2x^{1/4}h_2(z) \qquad v = tx^{-3/4}h_2(z) - 2x^{-1/4}h_1(z)$$
$$z = x^{-1/2}(t^2 - 4x). \tag{5.14}$$

Here the equation (4.7) reads:

$$\bar\eta \equiv (16x^2 + 8xt^2 + t^4)u_{tt} + (32x^2t + 8xt^3)u_{xt} + 16x^2t^2u_{xx}$$
$$+ 16xtu_t + (16x^2 + 12xt^2)u_x - 4xu = 0. \tag{5.15}$$

The $\mathscr{F}_E^*$ solutions are

$$u = a_1t(t^2 - 4x)^{-1/2} + a_2x(t^2 - 4x)^{-3/2} + b_1tx(t^2 - 4x)^{-5/2} + b_2(t^2 - 4x)^{1/2} \tag{5.16}$$

where a_i, b_i are arbitrary constants.

The family $\mathscr{F}_E$ is given by the functions (5.16) which are also solutions of

$$\tilde\eta \equiv (4x^2 - 3xt^2)u_x - t^3u_t + 2xu = 0 \tag{5.17}$$

obtained for $b_1 = b_2 = 0$.

In this case (5.15) is the invariant condition surface of a classical generalized symmetry of (2.5), and (5.17) is the invariant condition surface of a non-classical point symmetry of (2.5).

If we consider the system $\hat{\mathscr{S}}_p$

$$v_x = u_t \qquad v_t = xu_x - u$$

(particular case of (3.13) with $H = 1$ and $K_1 = K_2 = 0$), we recognize that the only potential symmetry is

$$\hat{\mathscr{X}}: \qquad \xi = 4xt \qquad \tau = 4x + t^2 \qquad \eta = -2v \qquad \phi = 2tv - 2xu.$$

The solutions $\mathscr{F}_E^*$ are as in (5.16), obtained using the previous conservative form, and the solutions $\mathscr{F}_E$ are obtained for $a_1 = a_2 = 0$ in (5.16).

Example 3.

As the last example we consider the quasilinear hyperbolic equation

$$u_{tt} = [f(u)u_x]_x \qquad f \in C^2(\mathbb{R}) \qquad f > 0 \tag{5.18}$$

which was extensively studied in [9]. If we consider the corresponding system $\mathscr{S}_p$

$$v_x = u_t \qquad v_t = f(u)u_x \tag{5.19}$$

obviously the first equation is a particular case of (3.13) with $H = 1$, $K_1 = K_2 = 0$ and it easy to check that for every choice of the arbitrary function f, (5.19) admits the point symmetry associated with the generators

$$\mathscr{X}: \qquad \xi = v + x \qquad \tau = u + t \qquad \eta = 0 \qquad \phi = 0 \tag{5.20}$$

which is a potential symmetry of (5.19).

The solutions of the invariant surface conditions of the vector field (5.20) are

$$u = h_1(z) \qquad v = h_2(z) \tag{5.21}$$

with the similarity variable implicitly defined by

$$(t + h_1)z - (x + h_2) = 0. \tag{5.22}$$

Here the equation (4.7) reads

$$\bar\eta \equiv u_{xx}u_t^2 + u_{tt}u_x^2 - 2u_xu_tu_{xt} = 0. \tag{5.23}$$

The system $\bar{\mathscr{S}}$ is

$$h_1'z+h_2'=0 \qquad h_1'f+h_2'z=0 \tag{5.24}$$

and the system $\bar{\bar{\mathscr{S}}}$ is

$$h_1''z^2+2zh_1'-fh_1''-(h_1')^2\dot{f}=0$$

$$(h_1'h_2''-h_2'h_1''+h_1h_1'')z^2+(2h_1h_1'-2h_1'h_2'-(h_1')^3\dot{f})z+(h_1')^2(h_2'-h_1)\dot{f} \tag{5.25}$$
$$+(h_2'h_1''-h_1h_1''-h_1'h_2''+2(h_1')^2)f=0$$

where

$$f=f(h_1) \qquad \text{and} \qquad \dot{f}=\left.\frac{\mathrm{d}f}{\mathrm{d}u}\right|_{u=h_1}.$$

If we assume $f=\log^2 u$, the solutions of $\bar{\mathscr{S}}$ are

$$\begin{aligned}
&\text{(i)} && h_1=\text{constant} && h_2=\text{constant} \\
&\text{(ii)} && h_1=\exp(z) && h_2=(1-z)\exp(z)+c \\
& && (2z-1)\exp(z)+tz+x-c=0 \\
&\text{(iii)} && h_1=\exp(-z) && h_2=-(1+z)\exp(-z)+c \\
& && (tz-x-c)\exp(z)+2z+1=0.
\end{aligned} \tag{5.26}$$

We can observe that the system $\bar{\bar{\mathscr{S}}}$ admits other solutions than (5.26), for example

$$h_1=\exp(z) \qquad h_2=c\exp(z) \qquad (z-c)\exp(z)+tz-x=0$$

which is not a solution of $\bar{\mathscr{S}}$.

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